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Homology and Completions of Groups

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INTRODUCTION

To what extent is a group determined by its homology? In other words, if A is a group and R is a commutative ring, can we reconstruct A from information about $H_* = H_*(A; R)$?

Two examples come to mind: Abelian groups and groups that Abelinanize to zero. Among the latter we can find nontrivial examples with $H_* \equiv 0$ such as $\langle a, b, c, d : a^{-1}ba = b^2,\ b^{-1}cb = c^2,\ c^{-1}dc = d^2,\ d^{-1}ad = a^2\rangle$.[1] On the other hand, if A is Abelian and $R = Q$, $H_1 = A \otimes Q$, and that is all we can recover. In other words we lose the Z-torsion subgroup.

These two examples set the tone of this paper: On the one hand we must give up the perfect part of the group. This amounts to assuming our groups are *residually nilpotent* [17, p. 349]. On the other hand H_* can only yield the non-Abelian generalization of $A \otimes R$, that is, the nilpotent R-completion $\hat{A}_R$ of A [16, II.3]. $\hat{A}_R$ is equipped with a map $\iota_A\colon A \to \hat{A}_R$ which is monic iff A is residually and A has no R-torsion in some sense.

Having settled for $\hat{A}_R(\iota_A$ injective), we find a partial answer to our question: A group homomorphism $h\colon A \to B$ is said to be an HR-equivalence if $h_*\colon H_q(A; R) \to H_q(B; R)$ is an isomorphism for $q = 1$ an epimorphism for $q = 2$. In [19] it is shown that if h is an HR-equivalence, it induces an isomorphism $\hat{A}_R \to \hat{B}_R$. This indicates that if our question has an answer at all, H_1 and H_2 are the only relevant groups. The following example, however, shows that isomorphy of H_1 and H_2 is not enough. An m-link is an embedding l of m circles $C_1, \ldots, C_m$ in S^3. The complement X of $\mathrm{Im}(l)$ is called the *complement* of l. Unless l is splittable, $\pi_2(X) = 0$ and so the group $G = \pi_1(X)$ has the following properties:

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[1] The example is due to Higman [13]. Such groups are called *perfect*. That $H_* \equiv 0$ is a result of Dyer and Vazquez [10].

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(i) $H_1(G; Z) = H_1(X) = Z^m$,

(ii) $H_2(G; Z) = \text{coker}(\eta\colon \pi_2(X) \to H_2(X)) = Z^{m-1}$, where η is the Hurewicz map.

But $\hat{G}_Z$ varies a lot from link to link. For the standard 2-link with linking number one $G = \hat{G}_Z = Z^2$. For all other 2-links G is not Abelian and $G \neq \hat{G}_Z$. Thus, some other information is needed.

The desired extra information is precisely the Massey coproducts $H_2 \to H_1 \otimes \cdots \otimes H_1$, which arise as differentials of the cobar construction spectral sequence of $B_*(A)$ (= reduced bar construction of the group algebra RA) or of any complex homotopic to H.

Roughly our main result is

(*) *If A and B are groups which embed in their R-completions, and if there exists an abstract isomorphism φ_q: $H_q(A; R) \to H_q(B; R)$ $(q = 1, 2)$ such that φ_* preserves the Massey coproducts, then φ induces an isomorphism $\hat{A}_R \to \hat{B}_R$.*

In other words, $\hat{A}_R$ can be reconstructed from H_1, H_2 and the Massey coproducts.

We now discuss a partial converse to (*). Over the integers we have an absolute theorem. Recall that a group G is parafree of rank r if it is residually nilpotent and if $\hat{G}_Z \approx \hat{F}_Z$ for some free group F of rank r.

If G is finitely generated and parafree then $H_2(G; Z) = 0$.

This is a conjecture due to Baumslag [3].

The result as stated requires some explanation: why finitely generated groups? If F is the free group of rank r [17, I], then $H_2(F; Z) = 0$ while an unpublished calculation of Bousfield, Dror, and Stammbach shows that the 2nd homology of the uncountable group $\hat{F}_Z$ is itself uncountable. But F and $\hat{F}_Z$ share, of course, the same completion.

For finitely generated torsion free groups, the converse of (*) holds provided R is a field.

It turns out we need nice (finitely generated and *pre-Abelian*) presentations for our groups. Cohen [9] proves *all* groups have pre-Abelian (although not necessarily finite) presentations. If a group G is finitely generated, a pre-Abelian presentation with finitely many generators exists [17, p. 141].

Relations with topology. (1) In [18] Milnor defines certain isotopy [19, Sect. 5] invariants $\bar{\mu}(i_1, \ldots, i_s)$ $(1 \leqslant i_j \leqslant m)$ for an m-link l. There are precisely the Massey coprodicts for the group $G = \pi_1(X)$. Let $\gamma_q(G)$ be the qth member of the lower central series [17, V] of G and $\gamma G = \bigcap \gamma_q G$. Then $\Gamma G = G/\gamma G$ is residually nilpotent. If ΓG is parafree all the $\bar{\mu}(i_1, \ldots, i_s)$ are zero. The converse however is not true because $H_2(\Gamma G)$ need not be zero (see (ii) above). In general we refer to the Massey coproducts as the Milnor $\bar{\mu}$-invariants of G [13].

(2) A group G is HR-local [4] if every HR-equivalence h: $A \to B$ induces

a bijection $\mathrm{Hom}(B, G) \to \mathrm{Hom}(A, G)$. Every group has an HR-localization $\bar{G}_R$ and a map $j_G\colon G \to \bar{G}_R$ which is an HR-equivalence. If G is nilpotent [21, VI; 22], then j_* is a homology localization, that is, $H_*(\bar{G}_R; Z) = H_*(G) \otimes R$ and $j_*(\alpha) = \alpha \otimes 1$. Further, $i_G = j_G$ and $\bar{G}_R = \hat{G}_R$. This is no longer true if G is not nilpotent (e.g., G free, $H_2(\hat{G}_Z) \neq 0$). We show that $\hat{G}_R$ still determines the homology of its finitely generated subgroups $A \supseteq G$ (that is, $H_*(A; R) \approx H_*(G; R)$). If P is a set of primes and $R = Z_P$, the Sullivan P-adic completion $\hat{K}$ of $K = K(G, 1)$ [22] contains spaces $X \supseteq K$ which almost are $K(A, 1)$'s for $G \subseteq A \subseteq \hat{G}_R$ in the sense that $H_2(X; R = H_2(G; R) \oplus [\mathrm{Im}(\eta) \otimes R]$ (η = Hurewicz map). For the not finitely generated subgroups of $\hat{G}_R$ presumably we must make use of the natural topology defined on $\hat{G}_R$. This last remark is due to Lyndon.

Hypotheses and notation. Unless otherwise specified we assume R is a field (Q, $Z/(p)$). If not, generally R is either Z or Z_p.

We always assume our groups over residually nilpotent; otherwise we could replace G by $\Gamma G = G/\bigcap \gamma_q(G)$. The group $\Gamma^q G = G/\gamma_q(G)$ is nilpotent of class $\leqslant q$. In [21] the notion of tensor product of a nilpotent group and R is defined. We repeat that definition in Section 3. For the time being, put $\Gamma_R^q G$ for $\Gamma^q G \otimes R$. The inclusion maps $\gamma_{q+1} G \to \gamma_q G$ induce epimorphisms $\pi_q\colon \Gamma_R^{q+1} G \to \Gamma_R^q G$ for all rings R. Then the system $\{\Gamma_R^q G, \pi_{q-1}\}$ is an inverse system. Let $\hat{G}_R$ be its limit and $\iota_G\colon G \to \hat{G}_R$ the map defined by the epimorphisms $\iota_q\colon G \to \Gamma^q G$. From the definitions, if $R = Z_P$, ι_G is monic iff G is residually nilpotent and has no π-torsion [21, V] for all primes $\pi \notin P$.

Similarly if $A = RG$ is the group R-algebra of G with augmentation ideal K, we assume $K^\omega = \bigcap K^n$ is zero. If R is a field and if G is residually nilpotent, then $K^\omega = 0$ [16]. However, for $R = Z$ this is not clear, at least not to the author. Emphasis should be made, therefore, on this essential hypothesis especially for Theorem 5 below.

1. Cobar Construction

1.1. Let G be a group and let R be a ring. Define $A = RG$, the group R-algebra with augmentation $\epsilon\colon A \to R$, diagonal $D\colon A \to A \otimes_R A$ (induced by the diagonal $G \to G \times G$) and fundamental ideal $K = \mathrm{Ker}\, \epsilon$. Occasionally we use $\mathscr{A}$, the complex defined by $\mathscr{A}_0 = A$, $\mathscr{A}_i = 0$ for $i \neq 0$, and the R-linear map $\varphi\colon A \to A$ given by $\varphi(a) = a - \epsilon(a)$.

Consider

$$\longrightarrow \mathscr{C}_3 \longrightarrow \mathscr{C}_2 \longrightarrow \mathscr{C}_1 \longrightarrow \mathscr{C}_0 \xrightarrow{\epsilon} R, \qquad (\mathscr{C}_*)$$

which is an A-resolution of R. Assume $\mathscr{C}_0 = A$. To compute $H_*(G; R)$ we use $\bar{\mathscr{C}}_* = \mathscr{C}_* \otimes_G R$. Then $\bar{\mathscr{C}}_0 = R$. We also assume for the applications that $\bar{\mathscr{C}}_1 = A$.

The cobar construction for RG is a double complex with a structure of DGA-algebra:

$$\mathscr{F}_{-p,q} = \sum \bar{\mathscr{C}}_{i_0} \otimes \cdots \otimes \bar{\mathscr{C}}_{i_p} \qquad \left(i_k > 0, \sum i_k = q+1\right)$$

with elements $[y_0 \mid \cdots \mid y_p]$ and differentials $d\colon \mathscr{F}_{-p,q} \to \mathscr{F}_{-p,q+1}$ and $D\colon \mathscr{F}_{-p,q} \to \mathscr{F}_{-p-1,q}$ induced by $d\colon \bar{\mathscr{C}}_n \to \bar{\mathscr{C}}_{n-1}$ and $D\colon \bar{\mathscr{C}}_n \to \sum_{i\neq 0,n} \bar{\mathscr{C}}_i \otimes \bar{\mathscr{C}}_{n-i}$, respectively. The grading here is not that of [5, Sect. 10] and it was introduced by Stallings in [19]. The advantage of this grading is twofold: (i) in the applications it is convenient to have $\mathscr{F}_{0,0} = \bar{\mathscr{C}}_1$, $\mathscr{F}_{01} = \bar{\mathscr{C}}_2$, (ii) $\mathscr{F}$ is the cotensor product $\mathscr{Y}(R) \,\square_{\bar{\mathscr{C}}}\, R$ [7] of the *unreduced* cobar construction and the grading of $\mathscr{F}$ corresponds to that of $\mathscr{Y}(R)$. Here the assumption $\bar{\mathscr{C}}_0 = R$ becomes essential (loc. cit.).

Define a map of DGA-algebras $\varphi\colon \mathscr{F} \to \mathscr{A}$ by

$$\begin{aligned} \varphi[x_0 \mid \cdots \mid x_p] &= (-1)^{p+1} \prod \varphi(x_i), && \text{if all} \quad x_i \in \bar{\mathscr{C}}_1, \\ &= 0 && \text{otherwise.} \end{aligned}$$

Then φ can be considered as a twisting cochain [5] in $C^*(\bar{\mathscr{C}}; A)$.

Consider the filtration

$$\Phi_{-n}\mathscr{F} = \sum_{p\geqslant n} \mathscr{F}_{-p,q};$$

Φ induces, *via* φ, a filtration Φ on A, namely, $\Phi_{-n}A = K^n$.

The calculations of Adams and Hilton [1] and Brown [5] easily extend to this case. $\bar{\mathscr{C}} \otimes \mathscr{F}$ is an acyclic complex and $\mathscr{C}$ is the twisted tensor product [5] $\bar{\mathscr{C}} \otimes_\varphi \mathscr{A}$. The map

$$(\mathrm{id}) \otimes \varphi\colon \bar{\mathscr{C}} \otimes \mathscr{F} \to \bar{\mathscr{C}} \otimes_\varphi \mathscr{A}$$

is a chain equivalence since both sides are acyclic. Theorem B of [6] insures that

LEMMA 1. *$\varphi\colon \mathscr{F} \to \mathscr{A}$ is a chain equivalence.*

Every double complex $\mathscr{F}_{-p,q}$ induces a spectral sequence $E^r_{-p,q}$ which does not necessarily converge finitely. Also, it does not follow that $E^r \Rightarrow \operatorname{gr}\mathscr{A} = \sum K^q/K^{q+1}$. However, since $\bigcap K^q = 0$ we have the following result.

THEOREM 2 (Stallings [20]). *If $K^\omega = 0$, then*

$$E^{p+1}_{-p,p} = E^\infty_{-p,p} = K^{p+1}/K^{p+2} \qquad \textit{for} \quad p \geqslant 0.$$

Proof. Define $\alpha\colon H_p(\Phi_{-q}\mathscr{F}) \to H_p(\Phi_{-q+1}\mathscr{F})$ by inclusion. Let $I_{p,q}$ be $\bigcap_k \alpha^k H_p(\Phi_{-p-k}\mathscr{F})$; there is a natural map $\alpha''\colon I_{p,q} \to I_{p,q-1}$ which for $p = 0$ is injective iff E^r converges to $\operatorname{gr}\mathscr{A}_0$ [8, XV. 2]. We prove that $I_{0q} = 0$ for all a: Let $z \in \sum_{i\geqslant q} \mathscr{F}_{-i,i}$ be a representative of an element in I_{0q}. Then z is homologous in $\mathscr{F}$ to a cycle in $\Phi_{-n}\mathscr{F}$ for any $n \geqslant p$. It follows that $\varphi(z) \in K^n$ $(n \geqslant p)$. Since φ is a chain equivalence and $K^\omega = 0$, z is a boundary in $\mathscr{F}$ and $I_{0q} = 0$.

For $s = q - p \geqslant 1$, $E^r_{-p,q}$ no longer converges to gr $\mathscr{A}_s = 0$.

The description of the differentials

$$d^r\colon E^r_{0,1} \to E^r_{-r+1,r-1}$$

is dual to that of the Massey products given in [12, p. 58] (see [11]). We refer to the d^r as the Massey coproducts. Observe that $E^r_{01} \subseteq H_2(G; R)$ and $E^r_{-p,q}$ is the quotient of the $(p + 1)$-fold tensor product of $H_1(G; R)$ by the images of the d^s $(s < r)$.

1.2. Suppose G has a finite presentation [17, I].

$$\langle x_1, \dots, x_n : r_1, \dots, r_m\rangle \tag{1}$$

and that the Abelianization of G is $Z^s \oplus \sum_{j=1}^t Z/(e_j)$ $(e_1 \mid \cdots \mid e_t)$. Then (1) can be changed to a presentation

$$\langle a_1, \dots, a_s, b_1, \dots, b_t, c_1, \dots, c_u\colon b_1^{e_1}B_1, \dots, b_t^{e_t}, c_1C_1, \dots, c_uC_u, D_1, \dots, D_v\rangle, \tag{2}$$

where, if Ψ is the free group in $\{a_i, b_j, c_k\}$, then the B_j, C_k, and D_l lie in $[\Psi, \Psi] = \gamma_2\Psi$. Such presentation is called *pre-Abelian* and [17, p. 141] if (1) is finite, we may choose u, v so that $s + t + u = n$, $t + u + v = m$.

Let L be a free group $\langle a_1, \dots, a_s\colon\rangle$ and $\mathscr{M}$ the ring of formal power series $R[[\alpha_1, \dots, \alpha_s]]$ in noncommutative variables α (Magnus ring [16]). Then the map $\mu\colon L \to \mathscr{M}$ defined by $\mu(a_i) = 1 + \alpha_i$ induces an R-algebra monomorphism $RL \to \mathscr{M}$. For $w \in L$, we write $\mu(w) = 1 + \sum \mu(i_1, \dots, i_p)\, \alpha_{i_1} \cdots \alpha_{i_p}$ $(1 \leqslant i_k \leqslant s)$. The coefficients $\mu(i_1, \dots, i_p)$ are the images of the Fox derivatives [17, p. 365].

$$(\partial^p w/\partial a_{i_1} \cdots \partial a_{i_p})^0 = \epsilon(\partial^p w/\partial^p a)$$

under the augmentation map $\epsilon\colon RL \to R$.

1.3. With the aid of (1) we may find an acyclic RG-complex $\mathscr{C}_*$ as in (1.1). For the applications we only need the two-skeleton of $\mathscr{C}_*$. This is described in [23] as follows:

Let $[\xi_i\}$ be in 1–1 correspondence with the x_i and $\{\mathfrak{x}_j\}$ in 1–1 correspondence with the r_j. Define

$$\mathscr{C}_0 = Ae, \qquad \epsilon(e) = 1,$$

$$\mathscr{C}_1 = \sum_{i=1}^n A\xi_i, \qquad d(\xi_i) = (x_i - 1)\, e,$$

and

$$\mathscr{C}_2 = \sum_{j=1}^m A\rho_j, \qquad d(\rho_j) = \sum (\partial r_j/\partial x_i)\, \xi_i\,.$$

Then it is easy to check $D: \mathscr{C}_2 \to \mathscr{C}_1 \otimes \mathscr{C}_1$ is given by

$$D(\rho_j) = \sum (\partial^2 r_j/\partial x_i\, \partial x_k)^0\, [\xi_i \mid \xi_k].$$

In general if if $c = \sum a_j \rho_j \in E_{01}^s$, then

$$d^s c = \sum a_j (\partial^{s+1} r_j/\partial x_{i_0} \cdots \partial x_{i_s})^0\, [\xi_{i_0} \mid \cdots \mid \xi_{i_s}] \tag{3}$$

(cf. [13, (2)].

1.4. If G is a group, we write $Q_p(G\colon R)$ for K^p/K^{p+1}, where $K = \ker(\epsilon\colon RG \to R)$. Q is a functor.

COROLLARY 3. *Let G be a group presented by* (1) *and W a set of words $w_1, \ldots, w_t$ in the x_i such that*

$$(\partial^s w_j/\partial x_{i_1} \cdots \partial x_{i_s})^0 = 0$$

for all $s \leqslant p$. Then if $H = G/\langle W \rangle$, the canonical map $Q_s(G) \to Q_s(H)$ is an isomorphism for all $s \leqslant p$.

If $\gamma_p(G)$ is the lower central series of G, $\sum (\gamma_p G/\gamma_{p+1} G) \otimes R$ has a structure of Lie algebra [2, 16] which we call L_G. If R is a field and U_G is the universal U_G is the universal enveloping algebra of L_G, the map $\psi\colon L_G \to Q_*(G;\, R)$ defined by $(g\gamma_{p+1}G) \to (g - 1)\, K^{p+1}$ for $g \in \gamma_p G$, induces an isomorphism $U_G \to Q_*(G)$ of graded R-algebras. This "Poincaré–Witt" theorem is due to Quillen (cf. [2, 21]).

As in (1), if F is the group $\langle x_1, \ldots, x_n\colon \rangle$ and S is the smallest normal subgroup of F generated by the $r_1, \ldots, r_m$, then it is well known [19, 21] that $H_2(G;\, Z) = (S \cap \gamma_2 F)/[F, S]$. As usual $[X, Y]$ is the smallest normal subgroup gnerated by the $[x, y] = xyx^{-1}y^{-1}$. Consider the filtration

$$\Phi_i H_2(G) = (S \cap \gamma_i F)/\{[F, S] \cap \gamma_i F\} = ((S \cap \gamma_i F)[F, S])/[F, S].$$

Then the corresponding graded object gr $H_2(G)$ is

$$\sum \gamma_i F/(\gamma_i F \cap [F, S]\, \gamma_{i+1} F).$$

On the other hand L_F is the free Lie algebra of rank n. We also have an epimorphism $L_F \to L_G$ with kernel $\mathfrak{S} = \sum (S \cap \gamma_i F)/(S \cap \gamma_{i+1} F)$. Under these conditions $H_2(L_G)$: $(\mathfrak{S} \cap [L_F, L_F])/[\mathfrak{S}, L_F]$ [8, p. 357] is isomorphic to

$$\sum \{(S \cap \gamma F)/\Pi_j [S \cap \gamma_{j-1} F, \gamma_{j-1} F])\}.$$

It follows that

LEMMA 4.

$$H_2(L_G) = \text{gr}\, H_2(G).$$

A good proof of this lemma, and many of the computations herein can be found in [11]. With this lemma we can compute the cobar construction for the algebra U_G. Over a field this does not differ from $Q(G)$; not so over Z.

2. The Term E_{01}^{∞}

2.1. For a group G we write $\Gamma^q G = G/\gamma_q G$ (cf. Introduction). A group is residually nilpotent iff $\gamma G = \bigcap \gamma_q G = 1$. As in the Introduction define $\hat{G} = \hat{G}_z = \text{inv lim}\{\Gamma^q G, \pi_q\}$. A residually nilpotent group G is parafree of rank s if we can find a free group $L = \langle x_1, ..., x_s : \rangle$ and isomorphisms $\nu_p: \Gamma^p L \to \Gamma^p G$ such that $\pi_p \nu_{p+1} = \nu_p \pi_p$. In particular $L \subseteq G \subseteq \hat{L}_Z$.

Theorem 5. *$H_2(G; Z) = 0$ for finitely generated parafree groups.*

Proof. By [19, 3.2] there is a monomorphism $h: L \to G$. Then, for some free group $\Lambda = \langle x_i, ..., x_s, x_{s+1}, ..., x_{s+t} : \rangle$ (t finite), there exist words B_j, C_k in γ_2 such that

$$\langle \Lambda : x_{s+1}B_1, ..., x_{s+t}B_t, C_1, ... \rangle$$

is a pre-Abelian (and finitely generated) presentation for G.

It will be convenient to assume $1 \leqslant i \leqslant s$ and $s + 1 \leqslant j \leqslant s + t$ throughout. Define $\psi: \Lambda \to L$ by $\psi(x_i) = x_i$, $\psi(x_j) = 1$. Clearly, ψ is a retraction of Λ on L. If ψ induces a map $\psi': G \to L$, then by [19, 3.4] ψ' must be an isomorphism. We use this fact to reason by contradiction: If we assume $H_2(G) \neq 0$, then we can prove that $\psi(B_j) = \psi(C_k) = 1$ so that ψ' is defined, but then $\psi_*': H_2(G) \to H_2(L) = 0$ is an isomorphism. This is a contradiction.

Now, the induced map $h: Q(L) \to Q(G)$ is an isomorphism since $U_L \approx U_G$ (see [2] for case $R = Z, L_G$ torsion free) and so $E^1_{-p,p} = E^{\infty}_{-p,p}$ because $Q_{p+1}(G) = Q_{p+1}(L) = E^1_{-p,p}$. As a result $E_{01}^{\infty} = H_2(G; Z)$ and $d^p: H_2(G) \to E^p_{-p,-1,p+1} = E^{\infty}_{-p-1,p+1}$ is zero for all $p \geqslant 2$. We study the consequences of this at the chain level in E^0. Let $S = \ker(\Lambda \to G)$. Assume $H_2 G = (S \cap \gamma_2 \Lambda)/[S, \Lambda] \neq 0$. Then one of the C_k, say $c = C_1$, does not belong to $[\Lambda, s]$ and $d^2 c$ is defined. As in (1.3) let $\mathscr{C}_1 = \sum Z\xi_l$ ($1 \leqslant l \leqslant s + t$) and $\mathscr{C}_2 = \sum Z\rho_p$ ($1 \geqslant p$). Since $d^2 c = 0$ in E^2, the chain $d^2 c$ must lie in $d^1 E^1_{-1,2}$. Now, $E^1_{-1,2}$ is a group with elements $f = \sum(e_{xr}\xi \otimes \rho + e'_{r'x'}\rho' \otimes \xi')$, $e, e' \in Z$ (we have eliminated the subscripts from x and r for typographical reasons. The r_p are the relations in (4)).

If $p \geqslant t + 1$, $(\partial r_p/\partial x_l)^0 = 0$ for all l in $[1, s + t]$ but for $p \leqslant t$, $(\partial r_p/\partial x_l)^0$ is 0 if $l \leqslant s$ and $\delta_{p,l-s}$ if $1 \leqslant p \leqslant t$ and $s \leqslant l \leqslant s + t$. Since $d^1 f = d^2 c$ for some f, it follows from (3) that $(\partial^2 c/\partial x_i\, \partial x_{i'})^0 = 0$. Similarly, the higher differentials $d^p c = d^1 f_p$ for f_p in $E^1_{-1,2}$, and so $(\partial^p c/\partial x_{i_1} \cdots \partial x_{i_p})^0 = 0$ ($1 \leqslant i_k \leqslant s$).

By [17, p. 395] if $\mathscr{L} = Z[[\xi_1, ..., \xi_{s+t}]]$ and $\mu(x_l) = 1 + \xi_l$ (cf. Sect. 1.2), then $\mu(c) \in \mathrm{M}$ does not involve terms in $\xi_{i_1} \cdots \xi_{i_p}$. It follows that $\psi(c) = 1$.

Let c' be the elements of Λ obtained by substituting the x_j by the x_jB_{j-s}. Then $c \equiv c' \bmod[\Lambda, S]$ and by the same argument $\psi(c') = 1$.

We use now the Taylor series $\mu(c)$ of c

$$1 + \sum (\partial^2 c/\partial x_i\, \partial x_j)^0\, (\xi_i\,, \xi_j) + \sum (\partial^2 c/\partial x_j\, \partial x_{j'})^0\, (\xi_j\,, \xi_{j'}) + \cdots, \tag{5}$$

where we write $(\xi_i\,, \xi_j)$ for $\xi_i\xi_j - \xi_j\xi_i$ in $\mathscr{L}$ and omit the $(\partial^2 c/\partial x_i\, \partial x_{i'})^0\, (\xi_i\,, \xi_{i'})$ since the coefficients vanish [17, p. 395].

Since $c \neq 1$, $(\partial^q c/\partial x_{k_1} \cdots \partial x_{k_p})^0 \neq 0$ for some q where at least one k_l is in $[s+1, s+t]$. Assume q is the smallest such number. Then by the formulas in [18, p. 295] we may assume $k_q = j_0$.

The series $\mu(c')$ is obtained from $\mu(c)$ by substituting ξ_j for $\mu(\xi_jB_{j-s}) - 1$ into it for all j. In other words

$$\xi_j = \sum (\partial^2 B_{j-s}/\partial x_{i'}\, \partial x_{i''})^0\, (\xi_{i'}\,, \xi_{i''}) + \sum (\partial^2 B_{j-s}/\partial x_i\, \partial x_{j''})^0\, (\xi_{i'}\xi_{j''}) + \cdots$$

and substitute in (5) to obtain

$$1 + \sum (\partial^2 c/\partial x_i\, \partial x_j)^0\, (\partial^2 B_{j-s}/\partial x_{i'}\, \partial x_{i''})^0\, (\xi_i\,, (\xi_{i'}\,, \xi_{i''})) + \cdots.$$

The coefficient for $(\xi_{k_1}\,, (\xi_{k_2}\,,..., (\xi_{i'}\,, \xi_{i''})\cdots)$ must be 0 since $\psi(c') = 1$. On the other hand this coefficient is $(\partial^q c/\partial x_{k_1} \cdots \partial x_{k_{q-1}}\, \partial x_{j_0})^0.\ (\partial^2 B_{j_0-s}/\partial x_{i'}\, \partial x_{i''})^0$ and its second factor must be zero since the first is not. By substituting x_j by x_jB_{j-s} into c' repeatedly we obtain $(\partial^p B_{j-s}/\partial x_{i_1} \cdots \partial x_{i_p})^0 = 0$ and $\psi(B_{j-s}) = 1$. From [19], ψ induces the desired isomorphism $G \to L$ which shows that the assumption $H_2(G; Z) \neq 0$ is contradictory.

Remark. If t is finite in (4) the number of x_j $(s + 1 \leqslant j \leqslant t)$ involved in the word obtained from c by substituting x_j by x_jB_{j-s} is finite. Even when we iterate that process the number is still $\leqslant t$. But if t were infinite this number could increase without bound and we would not be able to conclude $\psi(B_{j-s}) = 1$ for all j. Thus the finite generation of G is an essential hypothesis.

2.2. Assume now R is a field and G is a finitely generated residually nilpotent group. Let $E^i_{-p,q}$ be the associated cobar spectral sequence.

PROPOSITION 6. $E^\infty_{01} = 0$.

Proof. We assume $G \neq 1$, $G \neq \gamma_2 G$. This in turn means that if $K = \ker(RG \to R)$, then $K \neq 0$ and $K \neq K^\omega$, respectively.

By hypothesis, G has a presentation (2) with $s \neq 0$ (cf. Sect. 1.2); otherwise $H_1(G; Z) = \Gamma^2 G$ is finite and $K = K^2$, that is, either $K = 0$ or $K = K^\omega$.

As in Theorem 5, let Ψ be the free group in the a_i, b_j, c_k of (2). Define $\psi\colon \Psi \to L = \langle a_1 ,..., a_s : \rangle$ by $\psi(a_i) = a_i$, $\psi(b_j) = \psi(c_k) = 1$. Again we prove that if $E^\infty_{01} \subsetneq H_2(G; R)$ is nonzero $\psi(B_j) = \psi(C_k) = 1$ and ψ induces a retract a retract $\psi'\colon G \to L$. But then the $\psi'\colon \Gamma_R^n G \to \Gamma_R^n L$ are isomorphisms that commute with the $\pi\colon \Gamma_R^{n+1} \to \Gamma_R^n$. In conclusion, G is R-parafree. By Theorem 5 (with R coefficients) $H_2(G; R) = 0$ which contradicts $E_{01}^\infty \neq 0$.

3. Completions over Local Rings

3.1. We consider the case $R = Z_P$, where P is a set of primes [21, VII] For convenience we deal with the case $P = \varnothing$ (that is, $R = Q$) but general theorems can be stated and proved.

Let X, Y be two groups and let $\varphi_q\colon H_q(X) \to H_q(Y)$ ($q = 1, 2$; rational coefficients) be isomorphisms. We write $E^r_{-p,q}(X)$ for the cobar construction associated with QX. Clearly φ induces maps $E_{01}^k(X) \to E_{01}^r(Y)$ which we also call φ.

We say φ preserves the $\bar{\mu}$-invariants if the diagram

$$\begin{array}{ccc} E_{01}^r(X) & \xrightarrow{d^r} & E^r_{-r+1,r-1}(X) \\ {\scriptstyle\varphi}\downarrow & & {\scriptstyle\varphi}\downarrow \\ E_{01}^r(Y) & \xrightarrow{\bar{d}^r} & E^r_{-r+1,r-1}(Y) \end{array} \tag{6}$$

commutes.

Lemma 7. *Let G be a group as above, and let A be a finitely generated subgroup of $\hat{G}^A$ containing G. Then if $H_1(G) \to H_1(A)$ is onto, the inclusion $i\colon G \to A$ induces $\bar{\mu}$-invariant preserving isomorphisms $i_q\colon H_q(G) \to H_q(A)$ for $q = 1, 2$.*

Proof. Clearly $G \subseteq A \subseteq \hat{G}_R$ implies $H_1 G = H_1 A = H_1 \hat{G}_R$ so i_1 is an isomorphism.

(a) i_2 is monic. If $x \in \ker i_2$, assume $x \in E_{01}^r \subseteq H_2(G; R)$. Consider (6) with $X = G$, $Y = A$, $\varphi = i$. Clearly the diagram commutes by naturality and Coker $d^r = Q_{r+1}(G) =$ Coker $\bar{d}^r$ by Theorem 2. Then $d^r x = 0$, i.e., $x \in E_{01}^{s+1}$. By induction $x \in E_{01}^\infty$ which is zero by Proposition 6.

(b) Since Coker $d^r =$ coker $\bar{d}^r$, i_2 is epimorphic.

3.2. We now state the main result.

Theorem 8. *Let A, B be groups satisfying the hypotheses of this section (finitely generated, torsion free); then $\hat{A}_R \approx \hat{B}_R$ if and only if we have $\bar{\mu}$-invariant preserving isomorphisms $\varphi_q\colon H_q(A) \to H_q(B)$, $q = 1, 2$.*

Proof. If $\hat{A}^A \approx \hat{B}_R$ we may embed B in $\hat{A}_R$ in such a way that $A \subseteq B$. The result then follows from Lemma 7.

Conversely if we have μ-invariant preserving φ_q, then $\varphi\colon E_{01}^r(A) \to E_{01}^r(B)$

is an isomorphism for all r. Then $U_A \approx Q_*(A)$ is isomorphic to $U_B \approx Q_*(B)$. In particular, $L_A \approx L_B$, that is, $[\gamma_q(A)/\gamma_{q+1}(A)] \otimes Q \approx [\gamma_q(B)/\gamma_{q+1}(B)] \otimes Q$ for all q. We must show $\Gamma_Q{}^n A \approx \Gamma_Q{}^n B$ for all n.

First define $\Gamma_Q{}^n X$ for a group X. Since

$$0 \longrightarrow \gamma_n X/\gamma_{n+1} X \longrightarrow \Gamma^{n+1} X \xrightarrow{\pi_n} \Gamma^n X \longrightarrow 1, \qquad (7_n(X))$$

is a central extension, it is completely determined by a class $\chi_n(X) \in H^2(\Gamma^n X; \gamma_n X/\gamma_{n+1} X)$ [21, V]. If $\Gamma_Q{}^n X$ is defined, define $\Gamma_Q^{n+1} X$ by the central extension $0 \to [\gamma_n G/\gamma_{n+1} G] \otimes Q \to \Gamma_Q^{n+1} X \to \Gamma_Q{}^n X \to 1$ with class $\chi_n \otimes Q$ in

$$H^2(\Gamma_Q{}^n G; \gamma_n G/\gamma_{n+1} G)$$

(recall $\Gamma_Q{}^n X$ is a homology localization $\Gamma_Q{}^n X = (\Gamma^n X)_Q{}^-$, [21, VI]).

We show $\Gamma_Q^{n+1} A \approx \Gamma_Q^{n+1} B$ by induction. The strategy is to show classes $\chi_n(A) \otimes Q$ and $\chi_n(B) \otimes Q$ are equal.

Consider $\bar{\pi}$: $\Gamma_Q^{n+1} A \to \Gamma_Q{}^n A$. Then $\bar{\pi}$: $E^{n-2}_{-p,p}(\Gamma^{n+1} A) \to E^{n-2}_{-p,p}(\Gamma^n A)$ is a canonical isomorphism. Now d^n: $E_{01}^{n-1}(\Gamma^n A) \to E^{n-1}_{1-n,1-n}(\Gamma^n A)$ is an isomorphism since coker $\bar{d}^n = E^n_{1-n,n-1} = E^\infty_{1-n,n-1} = Q_n(\Gamma^n A) = 0$. On the other hand d^n: $E_{01}^{n-1}(\Gamma^{n+1} A) \to E^{n-1}_{1-n,n-1}(\Gamma^{n+1} A)$ has cokernel $Q_n(\Gamma^{n-1} A) = Q_n(A)$. By naturality, if we put $r = n - 1$, $X = \Gamma^{n+1} A$, $Y = \Gamma^n A$, and $\varphi = \pi$ in (6), we obtain a commutative diagram. Coker$(\pi_*$: $E_{01}^{n-1} \to E_{01}^{n-1}) = (\gamma_n A/\gamma_{n+1} A) \otimes R$. This follows from Lemma 4 (Hopf's formula). In fact, let F/S be a presentation for A. Then (with rational coefficients) $\gamma_n A/\gamma_{n+1} A = \gamma_n F/(S\gamma_{n+1} F \cap \gamma_n F)$. On the other hand,

$$E_{01}^{n-1}(\Gamma^n A) = (S\gamma_{n+1} F \cap \gamma_n F)/([S\gamma_n F, F] \cap \gamma_n F) = \gamma_n F/([S\gamma_n F, F] \cap \gamma_n F)$$

and coker π_* is $\gamma_n F/(S\gamma_{n+1} F \cap \gamma_n F)([S\gamma_n F, F] \cap \gamma_n F) = \gamma_n A/\gamma_{n+1} A$. Thus d^n determines π_*: $H_2(\Gamma^{n+1} A) \to H_2(\Gamma^n A)$ since $\bar{d}^n$ is an isomorphism and $\bar{\pi}_*$: $E^{n-1}_{1-n,n-1}(\Gamma^{n+1} A) \to E^{n-1}_{1-n,n-1}(\Gamma^n A)$ is also a canonical isomorphism.

Consider $H_2(\Gamma^{n+1} A) \xrightarrow{\pi_*} H_2(\Gamma^n A) \xrightarrow{\xi} \gamma_n A/\gamma_{n+1} A \to 0$ which by [19, (2 1)] is exact. By the above horrendous calculations ξ are determined by d^n since π_* is. On the other hand [21, II. 5], ξ is dual to $\chi_n(A) \otimes Q$ since $(7_n(A))$ is a central extension and $\mathrm{Exp}_Q(\Gamma^2 A, \gamma_n A/\gamma_{n+1} A) = 0$. Thus $(7_n(A))$ is determined by the spectral sequence and by naturality it induces isomorphisms $\Gamma_Q{}^n A \approx \Gamma_Q{}^n B$ which commute with π. It follows that $\hat{A}_Q \approx \hat{B}_Q$.

4. Topological Results

I. Let l: $mS^1 \to S^3$ be an embedding of the disjoint union mS^1 of circles S_i^1 for $i = 1, \ldots, m$. We may l is an m-link. Let X be the compact 3-manifold (the complement of l) obtained by removing open tubular neighborhoods

$T_i = S_i^1 \times D^2$ of $l(S_i^1)$ from S^3. Then $\partial X = \sum \partial T_i$. Let x_i (resp. y_i) be the meridian (resp. longitude) of the torus ∂T_i. The image (via the inclusion map $\partial T_i \subseteq X$) μ_i (resp. λ_i) of x_i (resp. y_i) in $\pi = \pi_1(X)$ is called an ith meridian (resp. longitude) of l.

If $G = \Gamma^p\pi$, then [18, Theorem 4] it has a presentation

$$\langle \mu_1, ..., \mu_m : [\mu_i, \lambda_i^{(p)}], \gamma_p L \rangle, \tag{8}$$

where L is the free group in the meridians μ_i and $\lambda_i^{(p)}$ are certain "parallels," i.e., words representing a loop in $T_i - l(S_i^1)$ homotopic to $l(S_i^1)$ in T_i.

By Corollary 3, $Q_p(\pi) \to Q_p(G)$ is an isomorphism. Let r_i be the relation $[\mu_i, \lambda_i^{(p)}]$ and let α_i be the generators of $\mathscr{C}_1$ ($\mathscr{C}_*$ is the complex associated with (8) as in Section 1.1). Write $[i_1 \mid \cdots \mid i_s]$ for $[\alpha_{i_1} \mid \cdots \mid \alpha_{i_s}]$. Then $d^s\colon E_{01}^s \to E_{-s,s}^s$ is

$$d^s(\rho_i) = \sum (\partial^s \lambda_k^{(p)}/\partial\mu_{k_1} \cdots \partial\mu_{k_s})^0 ([i \mid k_1 \mid \cdots \mid k_s] - [k_1 \mid \cdots \mid k_s \mid i]). \tag{9}$$

This follows from [18, (14), p. 294]. As usual ρ_i stands for the generator of $\mathscr{C}_2$ corresponding to r_i.

The $\bar{\mu}$-invariants of π as defined above and in [13] are obtained from the $(\partial^s\lambda_i^{(p)}/\partial\mu_{k_1} \cdots \partial\mu_{k_s})^0$ modulo the $(\partial^t\lambda_j^{(p)}/\partial\mu_{k_1} \cdots \partial\mu_{k_s})^0$, where $(j, l_1, ..., l_t)$ is obtained from $(i, k_1, ..., k_s)$ by deleting at least one entry and cyclically permuting the rest. In fact, by (9) in $Q_p(\pi)$

$$\sum \{(\partial^s\lambda_i^{(p)}/\partial\mu_{k_1} \cdots \partial\mu_{k_s})^0 - (\partial^s\lambda_{k_1}^{(p)}/\partial\mu_{k_2} \cdots \partial\mu_{k_s}\partial\mu_i)^0\} \; [i \mid k_1 \cdots \mid k_s] = 0.$$

These are the invariants introduced in [18]. Let now $L = \langle x_1, ..., x_m : \rangle$.

Corollary 9. *Let l be an m-link with group π. Then the homomorphism $h\colon L \to \pi$. defined by $h(x_i) = \mu_i$ induces an isomorphism $\hat{L}_Z \to \hat{\pi}_Z$ if and only if $H_2(\Gamma\pi) = 0$. In that case all the $\bar{\mu}$-invariants are zero.*

Remark. If all the $\bar{\mu} = 0$ it follows from (8) that $\Gamma^p L \approx \Gamma^p\pi$ for all p but it does *not* follow that this isomorphism is induced by a fixed $h\colon L \to \pi$. In fact in (8), $\lambda_i^{(p)}$ *depends on p* and so the homomorphism inducing $\Gamma^p L \approx \Gamma^p\pi$ need not be the same as that inducing $\Gamma^{p+1}L \approx \Gamma^{p+1}\pi$.

Proof of Corollary 9. If $H_2(\Gamma\pi) = 0$, $L \to \Gamma\pi$ satisfies the hypotheses of [19, (3.4)] and $\hat{L}_Z \approx (\Gamma\pi)_Z\hat{} \approx \hat{\pi}_Z$. The $\bar{\mu}$-invariants of π are induced by those of $\Gamma\pi$ which are zero. The converse is trivial.

II. The following is a very useful result in topology (cf. [14]): Let G be a group and let M be a G-group; that is, we are given a homomorphism $\psi\colon G \to \operatorname{Aut} M$. Let $D = M \times_\varphi G$, be the semidirect product of M and G. By [5] we may find a twisting cochain $\Psi \in C^*(\bar{\mathscr{B}}(RG); RG)$ such that $\bar{\mathscr{B}}(RD) = \bar{\mathscr{B}}(RG) \otimes_\Psi \bar{\mathscr{B}}(RM)$. Here $\mathscr{B}$ is the bar construction. We may assume $\bar{\mathscr{B}}(RG)_0 = R$, $\bar{\mathscr{B}}(RG)_1 = K = \ker(RG \to R)$ [16, p. 153].

As in Section 1, if $\mathscr{F}$ is the cobar construction associated to M

$$(\mathrm{id}) \otimes \varphi \otimes (\mathrm{id}): \bar{\mathscr{B}}(RG) \otimes_{\Psi'} \mathscr{F} \otimes_{\varphi} \bar{\mathscr{B}}(RM) \to \bar{\mathscr{B}}(RG) \otimes_{\Psi} (RM \otimes \bar{\mathscr{B}}(RM))$$

is a chain equivalence for a suitable twisting cochain Ψ' defined as in [5, Sect. 10]. Assume now R is a *field*.

LEMMA 10. $\mathrm{id} \otimes \varphi$: $\bar{\mathscr{B}}(RG) \otimes_{\Psi} \mathscr{F} \to \bar{\mathscr{B}}(RG) \otimes {}_{\psi}M$ *is a chain equivalence.*

Let M' be a G'-module and $f: G \to G'$ an HR-equivalence of residually nilpotent groups (cf. Introduction).

Suppose g: $M \to M'$ is a homomorphism over f.

THEOREM 11. *If* $(f, g)_*$: $H_q(G; M) \to H_q(G'; M')$ *is an isomorphism for* $q = 0, 1$, *then* g: $M/K^nM \to M'/K'^nM'$ *is an isomorphism for all* n.

Proof. $\mathscr{D}(G, M) = \mathscr{D} = \bar{\mathscr{B}}(RG) \otimes \mathscr{F}$. Then

$$\mathscr{D}_0 = \bar{\mathscr{F}}_0 = \sum_{n \geqslant 1} T^n H_0(G; M),$$

where T^n is the n-fold tensor product and $\bar{\mathscr{F}} = \mathscr{F} \otimes_G R$. In fact, $\mathscr{F}_0 = \sum_{n \geqslant 0} T^n M$. Tensoring over G with $\bar{\mathscr{B}}_0(RG) = R$ yields $\mathscr{D}_0 = \sum T^n(M \otimes_G R) = \sum T^n H_0(M; R)$. Similarly, $\mathscr{D}_1 = \bar{\mathscr{F}}_1 + K \otimes \bar{\mathscr{F}}_0$. The summands of $\bar{\mathscr{F}}_1$ are tensor products of $H_0(G; M)$ and one copy of $\bar{\mathscr{B}}_1(RA) \otimes_G R$. The differential $\bar{\mathscr{F}}_1 \to \bar{\mathscr{F}}_0$ is induced by that of $\bar{\mathscr{B}}_1(RA) \to \bar{\mathscr{B}}_0(RA)$ which is zero. The homology of $\mathscr{D}$ depends only on $K \otimes \bar{\mathscr{F}}_0$ since d: $\bar{\mathscr{F}}_2 \to \bar{\mathscr{F}}_1$ is an isomorphism (this follows from the fact that d: $\mathscr{F}_2 \to \mathscr{F}_1$ is an isomorphism by Lemma 1). By construction, the homology of $K \otimes \bar{\mathscr{F}}_0$ is $\sum T^n H_1(G; M)$ since R is a field.

Calculations similar to those leading to Theorem 2 show that if $E^{\infty}_{-p,q}$ is the spectral sequence of $\mathscr{D}$, then $E^r_{-p,p} = K^pM/K^{p+1}M$. Thus if f and g satisfy the hypotheses of the theorem, the spectral sequences associated to $\mathscr{D}(G, M)$ and $\mathscr{D}(G', M')$ are isomorphic.

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